

Recurrence Relations for Hypergeometric Functions of Unit Argument

By Stanisław Lewanowicz

Abstract. We show that the generalized hypergeometric function

$$P_n := {}_{p+3}F_{p+2} \left(\begin{matrix} -n, n + \lambda, a_p, 1 \\ b_{p+2} \end{matrix} \middle| 1 \right) \quad (n \geq 0)$$

satisfies a nonhomogeneous recurrence relation of order $p + \sigma$, where $\sigma = 0$ when ${}_{p+3}F_{p+2}(1)$ is balanced, and $\sigma = 1$ otherwise. Also, for

$$U_n := \frac{(c_{q+1})_n}{(d_q)_n (n + \lambda)_n} {}_{q+2}F_{q+1} \left(\begin{matrix} n + c_{q+2} \\ n + d_q, 2n + \lambda + 1 \end{matrix} \middle| 1 \right) \quad (n \geq 0)$$

a homogeneous recurrence relation of order $q + 1$ is given.

1. Notation. The generalized hypergeometric series

$$(1.1) \quad {}_{p+1}F_p \left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \left[\prod_{i=1}^{p+1} (a_i)_k z^k / k! \prod_{j=1}^p (b_j)_k \right],$$

where

$$(1.2) \quad (\alpha)_k := \Gamma(\alpha + k) / \Gamma(\alpha),$$

converges for $|z| < 1$. Further, this series is absolutely convergent for $|z| = 1$ if

$$(1.3) \quad \operatorname{Re} \left(\sum_{i=1}^{p+1} a_i - \sum_{j=1}^p b_j \right) < 0.$$

However, if any a_i is zero or a negative integer, then (1.1) always converges, since it terminates. If any b_j is 0 or a negative integer, the series is not defined. If $a_i = b_j$ for some i and j , $1 \leq i \leq p + 1$, $1 \leq j \leq p$, then ${}_{p+1}F_p$ reduces to ${}_pF_{p-1}$.

If in (1.1) $z = 1$, and the parameters are such that

$$(1.4) \quad 1 + \sum_{i=1}^{p+1} a_i = \sum_{j=1}^p b_j,$$

then the ${}_{p+1}F_p$ is said to be *balanced*.

Throughout, we employ a contracted notation. Given the function M , we define

$$M(a_{p+1}) := \prod_{i=1}^{p+1} M(a_i), \quad M(b_p) := \prod_{j=1}^p M(b_j).$$

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The symbol $1 + a_h - a_{p+1}^*$ stands for the set $1 + a_h - a_1, 1 + a_h - a_2, \dots, 1 + a_h - a_{h-1}, 1 + a_h - a_{h+1}, \dots, 1 + a_h - a_{p+1}$. Thus we may write (1.1) in abbreviated form

$${}_{p+1}F_p \left(\begin{matrix} a_{p+1} \\ b_p \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} [(a_{p+1})_k z^k / k! (b_p)_k].$$

For a treatment of the generalized hypergeometric functions, see any of the references [1], [4], [5].

2. Introduction. In this paper we derive new difference equations satisfied by the functions

$$(2.1) \quad P_n := {}_{p+3}F_{p+2} \left(\begin{matrix} -n, n + \lambda, a_p, 1 \\ b_{p+2} \end{matrix} \middle| 1 \right),$$

$$(2.2) \quad U_n := \frac{(c_{q+1})_n}{(d_q)_n (n + \lambda)_n} {}_{q+2}F_{q+1} \left(\begin{matrix} n + c_{q+2} \\ n + d_q, 2n + \lambda + 1 \end{matrix} \middle| 1 \right) \quad (n = 0, 1, \dots).$$

Notice that recursion relationships of the forms

$$(2.3) \quad \sum_{m=0}^{p+2} G_m(n; p + 2) \Phi_{n-m} = \gamma(n; p + 2),$$

$$(2.4) \quad \sum_{m=0}^{q+2} (-1)^m H_m(n; q + 2) \Psi_{n+m} = 0,$$

where $G_m(n; t)$, $H_m(n; t)$, $\gamma(n; t)$ are rational in n (see formulae (3.1), (3.2) and (3.11) below), are satisfied by (2.1) and (2.2), respectively. The relations are of order $p + 2$ and $q + 2$, respectively. This result can be deduced from some general theorems due to Wimp [8] (see also [4, vol. 2, p. 135 ff.]).

We show that (2.1) also obeys the difference equation

$$(2.5) \quad \sum_{m=0}^{p+\sigma} G_m(n; p + \sigma) \Phi_{n-m} = \gamma(n; p + \sigma),$$

of order $p + \sigma$, where $\sigma = 0$ for the balanced case,

$$\lambda + 2 + \sum_{i=1}^p a_i = \sum_{j=1}^{p+2} b_j,$$

and $\sigma = 1$ for the general case. Further, we show that (2.2) obeys the equation

$$(2.6) \quad \sum_{m=0}^{q+1} (-1)^m H_m(n; q + 1) \Psi_{n+m} = 0,$$

of order $q + 1$. Notice that the coefficients of Eqs. (2.3) and (2.5) are formed according to the same general rule (3.1). A similar statement is true for the coefficients of (2.4), (2.6) and for formula (3.2).

For applications, it is convenient to have available the functions which satisfy the same difference equation as does P_n (U_n , respectively), or the homogeneous form of this equation. A theorem, given in the next section, also provides information of this kind.

3. Results. Let p, q, n, m, t be integers ≥ 0 . Let λ, a_i ($i = 1, 2, \dots, p$), b_j ($j = 1, 2, \dots, p + 2$), c_k ($k = 1, 2, \dots, q + 2$), d_l ($l = 1, 2, \dots, q$) be complex constants. We define the functions

$$\begin{aligned}
 G_m(n; t) := & \frac{(n + 1 - m)_m (2n + \lambda - 2m)_{2m}}{(n + \lambda - m)_m (2n + \lambda - m - t)_m (n + b_{p+2} - 1)} \\
 & \times \left\{ \frac{1}{m!} (n - m - 1 + b_{p+2}) \right. \\
 (3.1) \quad & \times {}_{p+4}F_{p+3} \left(\begin{matrix} -m, 2n + \lambda - m - t, n - m + b_{p+2} \\ 2n + \lambda - 2m + 1, n - m - 1 + b_{p+2} \end{matrix} \middle| 1 \right) \\
 & + \frac{1}{\Gamma(m)} (2n + \lambda - m - t)(n - m + a_p) \\
 & \left. \times {}_{p+2}F_{p+1} \left(\begin{matrix} 1 - m, 2n + \lambda - t + 1, n - m + 1 + a_p \\ 2n + \lambda - 2m + 1, n - m + a_p \end{matrix} \middle| 1 \right) \right\},
 \end{aligned}$$

$$\begin{aligned}
 H_m(n; t) := & \frac{(2n + \lambda)_m (n + \beta + 1)_m}{(n + \lambda)_m} \\
 (3.2) \quad & \times \left\{ \frac{1}{m!} {}_{q+4}F_{q+3} \left(\begin{matrix} -m, n + 2n + \lambda, n + 1 + c_{q+2} \\ 2n + \lambda + q + 2, n + c_{q+2} \end{matrix} \middle| 1 \right) \right. \\
 & + \frac{(2n + \lambda + m)(n + d_q)}{\Gamma(m)(n + c_{q+2})} \\
 & \left. \times {}_{q+2}F_{q+1} \left(\begin{matrix} 1 - m, 2n + \lambda + m + 1, n + 1 + d_q \\ 2n + \lambda + q + 2, n + d_q \end{matrix} \middle| 1 \right) \right\}.
 \end{aligned}$$

Here $\beta + 1 = c_i$ for $i = q + 2$. We assume the parameters to be such that all expressions make sense. By applying a lemma of Wimp [8], we get the alternative forms for G_m, H_m :

$$\begin{aligned}
 G_m(n; t) = & \frac{(-1)^p (n + 1 - m)_m (2n + \lambda - 2m)(2n + \lambda - t + 1)_{t-1}}{(n + \lambda - m)_m (2n + \lambda - m - t)_m (n - 1 - b_{p+2})} \\
 & \times \left\{ \frac{1}{(t - m)!} (n + \lambda - t + 1 - b_{p+2}) \right. \\
 (3.3) \quad & \times {}_{p+4}F_{p+3} \left(\begin{matrix} m - t, 2n + \lambda - m - t, n + \lambda - t + 2 - b_{p+2} \\ 2n + \lambda - t + 1, n + \lambda - t + 1 + b_{p+2} \end{matrix} \middle| 1 \right) \\
 & + \frac{1}{\Gamma(t - m)} (2n + \lambda - m - t)(n + \lambda - t + 1 - a_p) \\
 & \left. \times {}_{p+2}F_{p+1} \left(\begin{matrix} m - t + 1, 2n + \lambda - m - t + 1, n + \lambda - t + 2 - a_p \\ 2n + \lambda - t + 1, n + \lambda - t + 1 - a_p \end{matrix} \middle| 1 \right) \right\},
 \end{aligned}$$

$$\begin{aligned}
 H_m(n; t) &= \frac{(-1)^q (2n + \lambda)_{q+2} (n + \beta + 1)_m}{(n + \lambda)_m (2n + \lambda + m)_{m+1} (n + c_{q+2})} \\
 &\times \left\{ \frac{(n + \lambda + m - c_{q+2})}{(t - m)!} \right. \\
 (3.4) \quad &\times {}_{q+4}F_{q+3} \left(\begin{matrix} m - t, 2n + \lambda + m, n + m + \lambda + 1 - c_{q+2} \\ 2n + \lambda + 2m + 1, n + \lambda + m - c_{q+2} \end{matrix} \middle| 1 \right) \\
 &+ \frac{1}{\Gamma(t - m)} (2n + \lambda + m)(n + \lambda + m + 1 - d_q) \\
 &\left. \times {}_{q+2}F_{q+1} \left(\begin{matrix} m - t + 1, 2n + \lambda + m + 1, n + m + \lambda + 2 - d_q \\ 2n + \lambda + 2m + 1, n + m + \lambda + 1 - d_q \end{matrix} \middle| 1 \right) \right\}.
 \end{aligned}$$

In particular, we have

$$\begin{aligned}
 G_0(n; t) &= H_0(n; t) \equiv 1, \\
 (3.5) \quad G_t(n; t) &= \frac{(-1)^p (n + 1 - t)_t (2n + \lambda - t + 1)_{t-1} (n + \lambda - t + 1 - b_{p+2})}{(n + \lambda - t)_t (2n + \lambda - 2t + 1)_{t-1} (n - 1 + b_{p+2})}, \\
 H_t(n; t) &= \frac{(-1)^q (2n + \lambda)_{q+2} (n + \beta + 1)_t (n + \lambda + t - c_{q+2})}{(n + \lambda)_t (2n + \lambda + t)_{t+1} (n + c_{q+2})}.
 \end{aligned}$$

Observe that

$${}_{r+2}F_{r+1} \left(\begin{matrix} -m, m + \alpha, 1 + \omega_r \\ k + \alpha, \omega_r \end{matrix} \middle| 1 \right) = 0 \quad (m, k = 0, 1, \dots)$$

for $m \geq k > r$ (see Luke [4, vol. 1, p. 114]). This shows that $G_m(n; t) = 0$ for $m \geq t + 1 > p + 2$, and $H_m(n; t) = 0$ for $m \geq t + 1 > q + 2$.

Let

$$(3.6) \quad G_m^*(n; t) := (n + \lambda - t)_t (2n + \lambda - 2t + 1)_{t-1} (n + b_{p+2} - 1) G_m(n; t).$$

Using (3.1), (3.3) and an identity from Luke [4, vol. 1, p. 42], one checks that

$$(3.7) \quad G_{t-m}^*(t - n - \lambda; t) = -G_m^*(n; t) \quad (m = 0, 1, \dots, t).$$

Similarly, for

$$\begin{aligned}
 (3.8) \quad H_m^*(n; t) &:= (n + \lambda)_m (n + m + \lambda - \beta)_{t-m} \\
 &\times (2n + \lambda + t + 1)_t (n + c_{q+2}) H_m(n; t),
 \end{aligned}$$

we have

$$(3.9) \quad H_{t-m}^*(-n - \lambda - t; t) = H_m^*(n; t) \quad (m = 0, 1, \dots, t).$$

We need the following result, which can be easily obtained with the aid of some formulae from Luke [4, vol. 1, p. 26]. If $m \geq r \geq 0$, m and r integers, then

$$(3.10) \quad {}_{r+2}F_{r+1} \left(\begin{matrix} -m, m + \alpha, 1 + \omega_r \\ r - h + \alpha, \omega_r \end{matrix} \middle| 1 \right) = \frac{(-1)^m m! \Gamma(r - h + \alpha)}{\Gamma(m + \alpha) \omega_r} s_h$$

for $h = 0, 1$, where

$$s_0 := 1, \quad s_1 := \frac{1}{2}(m)_2 + \frac{1}{2}(m-r+1)(m+r+2\alpha-2) + \sum_{i=1}^r \omega_i.$$

For n, t integers ≥ 0 , we define

$$(3.11) \quad \gamma(n; t) := \frac{(b_{p+2} - 1)(n + \lambda)_n}{(n + b_{p+2} - 1)(n + \lambda - t)_n}.$$

THEOREM. *The function*

$$(3.12) \quad P_n := {}_{p+3}F_{p+2} \left(\begin{matrix} -n, n + \lambda, a_p, 1 \\ b_{p+2} \end{matrix} \middle| 1 \right)$$

satisfies the difference equation

$$(3.13) \quad \sum_{m=0}^{p+\sigma} G_m(n; p + \sigma) \Phi_{n-m} = \gamma(n; p + \sigma) \quad (n \geq p + \sigma),$$

where the functions G_m and γ are defined by (3.1) (or (3.3)) and (3.11), respectively, and

$$(3.14) \quad \sigma := \operatorname{sgn} \left(\left| \lambda + 2 + \sum_{i=1}^p a_i - \sum_{j=1}^{p+2} b_j \right| \right).$$

Moreover, if

$$\operatorname{Re} \left(\lambda + \sum_{i=1}^p a_i - \sum_{j=1}^{p+2} b_j \right) < -2,$$

then another solution of (3.13) with $\sigma = 1$ is

$$(3.15) \quad \begin{aligned} Q_n &:= \frac{b_{p+2} - 1}{(n + 1)(n + \lambda - 1)(a_p - 1)} {}_{p+3}F_{p+2} \left(\begin{matrix} 2 - b_{p+2}, 1 \\ n + 2, 2 - n - \lambda, 2 - a_p \end{matrix} \middle| 1 \right) \\ &= 1 - {}_{p+3}F_{p+2} \left(\begin{matrix} 1 - b_{p+2}, 1 \\ n + 1, 1 - n - \lambda, 1 - a_p \end{matrix} \middle| 1 \right). \end{aligned}$$

Finally, if

$$\operatorname{Re} \left(\lambda + \sum_{i=1}^p a_i - \sum_{j=1}^{p+2} b_j \right) < -1,$$

then the functions

$$(3.16) \quad \begin{aligned} R_n^{(j)} &:= \frac{\Gamma(n + 1)\Gamma(n + \lambda - a_j)}{\Gamma(n + 1 + a_j)\Gamma(n + \lambda)} \\ &\quad \times {}_{p+2}F_{p+1} \left(\begin{matrix} 1 + a_j - b_{p+2} \\ 1 + a_j + n, 1 + a_j - n - \lambda, 1 + a_j - a_p^* \end{matrix} \middle| 1 \right) \end{aligned}$$

$(j = 1, 2, \dots, p),$

$$(3.17) \quad \begin{aligned} R_n^{(p+1)} &:= \frac{\Gamma(n + 1)\Gamma(1 - n - \lambda)\Gamma(2n + \lambda)\Gamma(n + a_p)}{\Gamma(n + b_{p+2})} \\ &\quad \times {}_{p+2}F_{p+1} \left(\begin{matrix} 1 - n - b_{p+2} \\ 1 - \lambda - 2n, 1 - n - a_p \end{matrix} \middle| 1 \right), \end{aligned}$$

$$(3.18) \quad R_n^{(p+2)} := \frac{\Gamma(1-n-\lambda)\Gamma(n+1)\Gamma(n+\lambda+1-b_{p+2})}{\Gamma(2n+\lambda+1)\Gamma(n+\lambda+1-a_p)} \\ \times {}_{p+2}F_{p+1} \left(\begin{matrix} n+\lambda+1-b_{p+2} \\ 2n+\lambda+1, 1+n+\lambda-a_p \end{matrix} \middle| 1 \right)$$

under the condition that no two of the quantities $-n, n+\lambda, a_i$ ($i = 1, 2, \dots, p$) differ by an integer or zero, and the functions

$$(3.19) \quad S_n^{(k)} := \frac{\Gamma(n+1)\Gamma(n+\lambda+1-b_k)}{\Gamma(n+b_k)\Gamma(n+\lambda)} \\ \times {}_{p+2}F_{p+1} \left(\begin{matrix} 1-b_k-n, 1-b_k+n+\lambda, 1-b_k+a_p \\ 1-b_k+b_{p+2}^* \end{matrix} \middle| 1 \right) \\ (k = 1, 2, \dots, p+2)$$

under the condition that no two of the parameters b_j ($j = 1, 2, \dots, p+2$) differ by an integer or zero, satisfy the difference equation

$$(3.20) \quad \sum_{m=0}^{p+1} G_m(n; p+1)\Phi_{n-m} = 0 \quad (n \geq p+1).$$

Remarks. 1°. It follows from a theorem in [4, vol. 2, p. 136], that (3.12), (3.15) satisfy Eq. (2.3), while the functions (3.16)–(3.19) are solutions of the homogeneous form of this equation. 2°. That the function (3.12) with $b_i = 1$ for $i = p+2$ satisfies a homogeneous equation of order $p+1$ (Eq. (3.20), in fact) follows from results of Wimp [10]. (Note that formulae (5), (6) of [10] should be corrected by multiplying the s th term of each sum by $1/s!$ (see [12]).) 3°. It can be conjectured that if no a_i is equal to any b_j , the functions (3.12), (3.15) do not satisfy a nontrivial equation of the form (3.13) of lower order than $p+\sigma$ or $p+1$, respectively, and none of the functions (3.16)–(3.19) satisfy an equation of the form (3.20) of lower order than $p+1$. (We assume that a_i, b_j are not interrelated.)

Proof. A. Putting the function (3.12) in the left-hand side of (3.13), turning the sum around, replacing n by $n+p+\sigma$, and performing some algebra, we get

$$(3.21) \quad \sum_{m=0}^{p+\sigma} G_{p+\sigma-m}(n+p+\sigma; p+\sigma)P_{n+m} = \gamma(n+p+\sigma; p+\sigma)(\phi_n + \psi_n),$$

where

$$\phi_n := \sum_{k=n}^{n+p+\sigma} \xi_k, \quad \psi_n := \frac{(n+p+\sigma)(n+\lambda)(a_p)^{n+p+\sigma-1}}{b_{p+2}-1} \sum_{k=n} \eta_k, \\ \xi_k := \frac{(-1)^k (-n-p-\sigma)_k (n+\lambda)_k (b_{p+2})_k}{k! (b_{p+2}-1)_k (k+\lambda)_k} \\ \times {}_{p+4}F_{p+3} \left(\begin{matrix} k-n-p-\sigma, k+n+\lambda, k+b_{p+2} \\ 2k+\lambda+1, k-1-b_{p+2} \end{matrix} \middle| 1 \right) \\ \times {}_{p+3}F_{p+2} \left(\begin{matrix} -k, k+\lambda, a_p, 1 \\ b_{p+2} \end{matrix} \middle| 1 \right) \quad (k \geq 0),$$

$$\eta_k := \frac{(-1)^k (1 - n - p - \sigma)_k (n + \lambda + 1)_k (1 + a_p)_k}{k! (k + \lambda)_k (a_p)_k} \\ \times {}_{p+2}F_{p+1} \left(\begin{matrix} k - n - p - \sigma + 1, k + n + \lambda + 1, k + 1 + a_p \\ 2k + \lambda + 1, k + a_p \end{matrix} \middle| 1 \right) \\ \times {}_{p+3}F_{p+2} \left(\begin{matrix} -k, k + \lambda, a_p, 1 \\ b_{p+2} \end{matrix} \middle| 1 \right) \quad (k \geq 0).$$

Let

$$f_n := {}_{p+3}F_{p+2} \left(\begin{matrix} -n - p - \sigma, n + \lambda, a_p, 1 \\ b_{p+2} - 1 \end{matrix} \middle| 1 \right).$$

By applying a theorem in [4, vol. 2, p. 7], we have

$$(3.22) \quad f_n = \sum_{k=0}^{n+p+\sigma} \xi_k = \phi_n + \rho_n,$$

where

$$\rho_n := \sum_{k=0}^{n-1} \xi_k.$$

Similarly, for

$$f_n^* := {}_{p+3}F_{p+2} \left(\begin{matrix} 1 - n - p - \sigma, n + \lambda + 1, 1 + a_p, 1 \\ b_{p+2} \end{matrix} \middle| 1 \right) \\ = \frac{b_{p+2} - 1}{(n + p + \sigma)(n + \lambda)a_p} (1 - f_n),$$

by virtue of the same theorem, we obtain

$$f_n^* = \sum_{k=0}^{n+p+\sigma-1} \eta_k,$$

which implies

$$(3.23) \quad 1 - f_n = \psi_n + \tau_n,$$

where

$$\tau_n := \frac{(n + p + \sigma)(n + \lambda)a_p}{b_{p+2} - 1} \sum_{k=0}^{n-1} \eta_k.$$

Applying (3.10), we get for $k = 0, 1, \dots, n - 1$, the equations

$${}_{p+4}F_{p+3} \left(\begin{matrix} -h, h + \mu, k + b_{p+2} \\ \mu + p + \sigma + 1, k - 1 - b_{p+2} \end{matrix} \middle| 1 \right) = \frac{(-1)^h h! \Gamma(2k + \lambda + 1)}{\Gamma(k + n + \lambda)(k - 1 + b_{p+2})} y_{k\sigma}, \\ {}_{p+2}F_{p+1} \left(\begin{matrix} 1 - h, h - 1 + \nu, k + 1 + a_p \\ \nu + p + \sigma - 1, k + a_p \end{matrix} \middle| 1 \right) = \frac{(-1)^{h-1} (h - 1)! \Gamma(2k + \lambda + 1)}{\Gamma(k + n + \lambda + 1)(k + a_p)} z_{k\sigma},$$

where

$$h := n + p + \sigma - k, \quad \mu := 2k + \lambda - p - \sigma, \quad \nu := \mu + 2, \\ y_{k1} := z_{k1} := 1, \quad y_{k0} := w_k + \sum_{i=1}^{p+2} b_i, \quad z_{k0} := w_k + \lambda + 2 + \sum_{i=1}^p a_i, \\ w_k := (k - 1)(p + 2) + \frac{1}{2}(n + p - k)_2 + \frac{1}{2}(n - k - 1)(n + 3k + 2\lambda).$$

Hence

$$\rho_n = \delta \sum_{k=0}^{n-1} \omega_k y_{k\sigma}, \quad \tau_n = -\delta \sum_{k=0}^{n-1} \omega_k z_{k\sigma},$$

where

$$\delta := \frac{(-1)^{p+n+\sigma} (p+n+\sigma)!}{\Gamma(n+\lambda)(b_{p+2}-1)},$$

$$\omega_k := \frac{(-1)^k (2k+\lambda)\Gamma(k+\lambda)}{k!} {}_{p+3}F_{p+2} \left(\begin{matrix} -k, k+\lambda, a_p, 1 \\ b_{p+2} \end{matrix} \middle| 1 \right)$$

($k = 0, 1, \dots, n-1$).

As, in view of the definition (3.14), $y_{k\sigma} = z_{k\sigma}$ for $\sigma = 0, 1$ and $k = 0, 1, \dots, n-1$, we obtain $\rho_n = -\tau_n$. This, together with (3.21)–(3.23), implies that the function (3.12) satisfies Eq. (3.13).

B. A similar argument shows that the function (3.15) satisfies Eq. (3.13) with $\sigma = 1$. Putting \mathcal{Q}_n in the left-hand side of (3.13), we get after some algebra

$$(3.24) \quad \sum_{m=0}^{p+1} G_{p+1-m}(n+p+1; p+1)\mathcal{Q}_{n+m} = \gamma(n+p+1; p+1)(\phi_n + \psi_n),$$

where

$$(3.25) \quad \phi_n := (n+\lambda+1-b_{p+2})\kappa_n$$

$$\times \sum_{m=0}^{p+1} \mu_m(n) {}_{p+4}F_{p+3} \left(\begin{matrix} -m, m+2n+\lambda, n+\lambda+2-b_{p+2} \\ 2n+\lambda+p+2, n+\lambda+1-b_{p+2} \end{matrix} \middle| 1 \right),$$

$$(3.26) \quad \psi_n := (n+\lambda+1-a_p)\kappa_n$$

$$\times \sum_{m=0}^p \mu_m(n+1) {}_{p+2}F_{p+1} \left(\begin{matrix} -m, m+2n+\lambda+2, n+\lambda+2-a_p \\ 2n+\lambda+p+2, n+\lambda+1-a_p \end{matrix} \middle| 1 \right),$$

$$\mu_m(n) := \frac{\Gamma(m+n+\lambda-1)\Gamma(m+2n+\lambda)(2m+2n+\lambda)}{m!\Gamma(m+n+2)}$$

$$\times {}_{p+3}F_{p+2} \left(\begin{matrix} 2-b_{p+2}, 1 \\ m+n+2, 2-n-\lambda-m, 2-a_p \end{matrix} \middle| 1 \right),$$

$$\kappa_n := \frac{\Gamma(n+p+2)}{(1-a_p)\Gamma(n+\lambda)\Gamma(2n+\lambda+p+2)}.$$

Notice the use of the second form, i.e. (3.3), for G_m . By applying a theorem from [4, vol. 2, p. 5], and making use of (3.10), we obtain the equations

$$\phi_n = \frac{2-b_{p+2}}{(1-a_p)(n+\lambda-1)(n+p+2)} g_n^* - \rho_n = 1 - g_n - \rho_n, \quad \psi_n = g_n + \rho_n,$$

where

$$g_n := {}_{p+3}F_{p+2} \left(\begin{matrix} 2 - b_{p+2}, 1 \\ n + p + 2, 1 - n - \lambda, 1 - a_p \end{matrix} \middle| 1 \right),$$

$$g_n^* := {}_{p+3}F_{p+2} \left(\begin{matrix} 3 - b_{p+2}, 1 \\ n + p + 3, 2 - n - \lambda, 2 - a_p \end{matrix} \middle| 1 \right),$$

$$\rho_n := \Gamma(2n + \lambda + p + 2) \kappa_n \sum_{m=p+2}^{\infty} (-1)^m \frac{m! \mu_m(n)}{\Gamma(m + 2n + \lambda)}.$$

Hence (3.24) is equivalent to (3.13) with $\Phi_n = Q_n$ and $\sigma = 1$.

C. We check that the functions (3.16) obey Eq. (3.20). Let j be any integer from the set $\{1, 2, \dots, p\}$. Proceeding as in part B of the proof, we get

$$(3.27) \quad \sum_{m=0}^{p+1} G_{p+1-m}(n + p + 1; p + 1) R_{n+m}^{(j)} = \gamma(n + p + 1; p + 1)(\phi_n + \psi_n),$$

where ϕ_n and ψ_n are as given in (3.25), (3.26), in which, however, the functions κ_n and $\mu_m(n)$ are defined anew by

$$\kappa_n := \frac{\Gamma(n + p + 2)}{(1 - b_{p+2})\Gamma(n + \lambda)\Gamma(2n + \lambda + p + 2)},$$

$$(3.28) \quad \mu_m(n) := \frac{\Gamma(m + n + \lambda - a_j)\Gamma(m + 2n + \lambda)(2m + 2n + \lambda)}{m!\Gamma(m + n + 1 - a_j)}$$

$$\times {}_{p+2}F_{p+1} \left(\begin{matrix} 1 + a_j - b_{p+2} \\ 1 + a_j + n + m, 1 + a_j - n - \lambda - m, 1 + a_j - a_p^* \end{matrix} \middle| 1 \right).$$

Applying the above-mentioned theorem from [4, vol. 2, p. 5], and then using identity (3.10), we may write

$$(3.29) \quad \phi_n = \kappa_n(f_n - \rho_n), \quad \psi_n = \kappa_n(\omega_n - f_n + \rho_n),$$

where

$$f_n := \frac{(1 + a_j - b_{p+2})\Gamma(n + \lambda - a_j)\Gamma(2n + \lambda + p + 2)}{\Gamma(n + p + 2 + a_j)}$$

$$\times {}_{p+2}F_{p+1} \left(\begin{matrix} 2 + a_j - b_{p+2} \\ 1 + a_j - n - \lambda, 2 + a_j + n + p, 1 + a_j - a_p^* \end{matrix} \middle| 1 \right),$$

$$\rho_n := \Gamma(2n + \lambda + p + 2) \sum_{m=p+2}^{\infty} (-1)^m \frac{m! \mu_m(n)}{\Gamma(m + 2n + \lambda)},$$

$$\omega_n := (n + \lambda + 1 - a_p)$$

$$\times \sum_{m=0}^{\infty} \frac{(2m + 2n + \lambda + 2)\Gamma(m + 2n + \lambda + 2)\Gamma(m + n + \lambda + 1 - a_j)}{m!\Gamma(m + n + 2 + a_j)}$$

$$\times {}_{p+2}F_{p+1} \left(\begin{matrix} -m, m + 2n + \lambda + 2, n + \lambda + 2 - a_p \\ 2n + \lambda + p + 2, n + \lambda + 1 - a_p \end{matrix} \middle| 1 \right).$$

Now, applying the identity

$$\sum_{m=0}^{\infty} \frac{(2m + \gamma)(-\mu)_m}{m!(m + \gamma)_{\mu+1}} {}_{p+2}F_{p+1} \left(\begin{matrix} -m, m + \gamma, c_p - \mu \\ \alpha, d_p - \mu \end{matrix} \middle| 1 \right) = \frac{(d_p)_{-\mu}}{(\alpha)_{\mu}(c_p)_{-\mu}},$$

which can be deduced from another theorem in [4, vol. 2, p. 1], we conclude, after making an identification of the parameters ($\gamma = 2n + \lambda + 2$, $\alpha = 2n + \lambda + p + 2$, $\mu = a_j - n - \lambda - 1$, $c_i = 1 + a_j - a_i$, $d_i = a_j - a_i$ for $i = 1, 2, \dots, p$), that $\omega_n = 0$. The result then follows from (3.27) and (3.29).

D. We show that the function $R_n^{(p+2)}$, defined in (3.18), obeys (3.20). Putting (3.18) in the left-hand side of (3.20) yields

$$(3.30) \quad \sum_{m=0}^{p+1} G_{p+1-m}(n + p + 1; p + 1) R_{n+m}^{(p+2)} = \gamma(n + p + 1; p + 1)(-1)^n c \kappa_n (\phi_n^* + \psi_n^*),$$

where c is a constant, κ_n is given by (3.28), and

$$\begin{aligned} \phi_n^* &:= (n + \lambda + 1 - b_{p+2}) \\ &\quad \times \sum_{m=0}^{p+1} v_m(n) {}_{p+4}F_{p+3} \left(\begin{matrix} -m, m + 2n + \lambda, n + \lambda + 2 - b_{p+2} \\ 2n + \lambda + p + 2, n + \lambda + 1 - b_{p+2} \end{matrix} \middle| 1 \right), \\ \psi_n^* &:= -(n + \lambda + 1 - a_p) \\ &\quad \times \sum_{m=0}^p v_m(n+1) {}_{p+2}F_{p+1} \left(\begin{matrix} -m, m + 2n + \lambda + 2, n + \lambda + 2 - a_p \\ 2n + \lambda + p + 2, n + \lambda + 1 - a_p \end{matrix} \middle| 1 \right), \\ v_m(n) &:= \frac{(-1)^m (2m + 2n + \lambda) \Gamma(m + 2n + \lambda) \Gamma(m + n + \lambda + 1 - b_{p+2})}{m! \Gamma(2m + 2n + \lambda + 1) \Gamma(m + n + \lambda + 1 - a_p)} \\ &\quad \times {}_{p+2}F_{p+1} \left(\begin{matrix} m + n + \lambda + 1 - b_{p+2} \\ 2m + 2n + \lambda + 1, m + n + \lambda + 1 - a_p \end{matrix} \middle| 1 \right). \end{aligned}$$

By virtue of the theorem mentioned in part A of the proof (see [4, vol. 2, p. 7]), and using identity (3.10), we get

$$\phi_n^* = -\psi_n^* = f_n - \rho_n,$$

where

$$\begin{aligned} f_n &:= \frac{\Gamma(n + \lambda + 2 - b_{p+2})}{\Gamma(n + \lambda + 1 - a_p)} {}_{p+2}F_{p+1} \left(\begin{matrix} n + \lambda + 2 - b_{p+2} \\ 2n + \lambda + p + 2, h + \lambda + 1 - a_p \end{matrix} \middle| 1 \right), \\ \rho_n &:= \Gamma(2n + \lambda + p + 2) \sum_{m=p+2}^{\infty} (-1)^m \frac{m! v_m(n)}{\Gamma(m + 2n + \lambda)}. \end{aligned}$$

Hence the result follows from (3.30).

E. The difference equation

$$(3.31) \quad \sum_{m=0}^{p+1} G_m^*(n; p + 1) \Phi_{n-m} = 0,$$

where the coefficients G_m^* are defined by (3.6), is obviously equivalent to Eq. (3.20). Because of the symmetry property (3.7), Eq. (3.31) is invariant under the change of variable $n := -n - \lambda$. Thus $R_n^{(p+1)} = \text{const} \cdot R_{-n-\lambda}^{(p+2)}$ must also be a solution of (3.20).

F. As it has already been remarked, the functions (3.16)–(3.19) satisfy the difference equation

$$\sum_{m=0}^{p+2} G_m(n; p+2) \Phi_{n-m} = 0$$

of order $p + 2$, and so any $p + 3$ of them are linearly dependent. Thus the functions $S_n^{(k)}$ ($k = 1, 2, \dots, p + 2$) satisfy Eq. (3.20). This completes the proof of the theorem.

As a corollary to the above results, we give a difference equation satisfied by a family of functions, including U_n as defined in (2.2).

COROLLARY. *Let n, q be integers ≥ 0 . Let β, λ, c_i ($i = 1, 2, \dots, q + 1$), d_j ($j = 1, 2, \dots, q$) be complex constants such that none of the quantities $\beta + 1, \lambda, c_i, d_j$ are negative integers or zero and that*

$$\text{Re} \left(\sum_{i=1}^{q+2} c_i - \sum_{j=1}^q d_j - \lambda \right) < 1,$$

where $c_{q+2} = \beta + 1$ for $i = q + 2$. Then the function

$$(3.32) \quad U_n := \frac{(c_{q+1})_n}{(d_q)_n (n + \lambda)_n} {}_{q+2}F_{q+1} \left(\begin{matrix} n + c_{q+2} \\ n + d_q, 2n + \lambda + 1 \end{matrix} \middle| 1 \right)$$

satisfies the difference equation

$$(3.33) \quad \sum_{m=0}^{q+1} (-1)^m H_m(n; q+1) \Psi_{n+m} = 0,$$

the coefficients H_m being given by (3.2) or (3.4). Further, if no two of the parameters $n, -n - \lambda, d_j$ ($j = 1, 2, \dots, q$) differ by an integer or zero, then the functions

$$(3.34) \quad V_n := \frac{\omega_n \Gamma(n + \lambda + 1 - d_q)}{\Gamma(n + \lambda + 1 - c_{q+2}) \Gamma(1 - \lambda - 2n)} \times {}_{q+2}F_{q+1} \left(\begin{matrix} c_{q+2} - n - \lambda \\ d_q - n - \lambda, 1 - \lambda - 2n \end{matrix} \middle| 1 \right),$$

$$(3.35) \quad W_n^{(j)} := \frac{\omega_n}{\Gamma(2 - d_j - n) \Gamma(n + \lambda + 2 - d_j)} \times {}_{q+2}F_{q+1} \left(\begin{matrix} 1 + d_j - c_{q+2} \\ 2 - d_j - n, n + \lambda + 2 - d_j, 1 - d_j + d_q^* \end{matrix} \middle| 1 \right) \quad (j = 1, 2, \dots, q)$$

are also solutions of (3.33). Here

$$\omega_n := (2n + \lambda) \Gamma(n + \lambda) / \Gamma(n + \beta + 1).$$

Finally, if no two of the parameters c_i ($i = 1, 2, \dots, q + 2$) differ by an integer or zero, then the functions

$$(3.36) \quad Z_n^{(k)} := \frac{(-1)^n \omega_n(c_k)_n}{\Gamma(n + \lambda + 1 - c_k) \Gamma(1 + c_k - c_{q+2})} \\ \times {}_{q+2}F_{q+1} \left(\begin{matrix} n + c_k, -n - \lambda + c_k, 1 - d_q + c_k \\ 1 + c_k - c_{q+2}^* \end{matrix} \middle| 1 \right) \\ (k = 1, 2, \dots, q + 2)$$

also satisfy Eq. (3.33).

Remarks. 1°. By virtue of a theorem in [8], the functions (3.32), (3.34)–(3.36) are solutions of (2.4). 2°. Making the substitution

$$\Psi_n^* := (\lambda - \beta)_n \Psi_n / (\lambda)_n$$

in (3.33) yields the equation

$$\sum_{m=0}^{q+1} (-1)^m H_m^*(n; q + 1) \Psi_{n+m}^* = 0,$$

H_m^* being defined by (3.8).

Proof. In (3.16)–(3.20), let $p := q$, $a_i := \lambda + 1 - d_i$, $b_j := \lambda + 1 - c_j$. Then we have

$$R_n^{(j)} = C^{(j)} \tau_n W_n^{(j)} \quad (j = 1, 2, \dots, p), \\ R_n^{(p+1)} = C^{(p+1)} \tau_n V_n, \quad R_n^{(p+2)} = C^{(p+2)} \tau_n U_n, \\ S_n^{(k)} = D^{(k)} \tau_n Z_n^{(k)} \quad (k = 1, 2, \dots, p + 2)$$

for some constants $C^{(j)}$, $D^{(k)}$, and

$$\tau_n := (-1)^n \Gamma(n + 1) \Gamma(n + \beta + 1) / (2n + \lambda) \Gamma^2(n + \lambda).$$

Now, in (3.20) make the substitution $\Phi_n := \tau_n \Psi_n$, turn the sum around and replace n by $n + p + 1$. The result is

$$\sum_{m=0}^{q+1} \tau_{n+m} G_{q+1-m}(n + q + 1; q + 1) \Psi_{n+m} = 0.$$

However, a simple calculation shows that the identity

$$\tau_{n+m} G_{q+1-m}(n + q + 1; q + 1) = \rho_n (-1)^m H_m(n; q + 1)$$

holds for some rational function ρ_n (cf. (3.2), (3.3)). Thus Ψ_n satisfies (3.33).

4. Examples. In this final section we illustrate the application of the theorem by taking some examples of problems in which functions of the type (2.1) or (2.2) occur. These problems were originally solved using different methods. We show that the solutions can be readily obtained by applying the results of Section 3. Also, there are new implications in some places, which seem to be interesting.

A. *Beta integral of the Jacobi function.* In [2], the integral

$$(4.1) \quad m_n := \int_0^1 x^{a-1} (1-x)^{b-1} R_n^{(\alpha, \beta)}(x) dx, \quad \operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0,$$

is examined. Here

$$(4.2) \quad R_n^{(\alpha, \beta)}(x) := \frac{(\alpha + 1)_n}{\Gamma(n + 1)} {}_2F_1 \left(\begin{matrix} -n, n + \lambda \\ \alpha + 1 \end{matrix} \middle| 1 - x \right) \quad (\lambda = \alpha + \beta + 1)$$

is the Jacobi function. If n is an integer ≥ 0 , then (4.2) is the n th shifted Jacobi polynomial. (4.1) is called a beta integral of the Jacobi function. It can be shown that

$$m_n = \frac{(\alpha + 1)_n B(a, b)}{\Gamma(n + 1)} {}_3F_2 \left(\begin{matrix} -n, n + \lambda, b \\ \alpha + 1, a + b \end{matrix} \middle| 1 \right).$$

The theorem of Section 3 yields the second-order recurrence relation

$$\begin{aligned} & n(n + \lambda - 1)(2n + \lambda - 3)(n + a + b - 1)m_n \\ & + (2n + \lambda - 3) {}_3[u(n) - u(n - 1) - n - \lambda + b + 1]m_{n-1} \\ & - (n + \alpha - 1)(n + \beta - 1)(2n + \lambda - 1)(n + \lambda - a - b - 1)m_{n-2} = 0, \\ & u(n) := (n + \beta)(n + \lambda - 1)(n + \lambda - a - b)/(2n + \lambda - 1). \end{aligned}$$

In a special case, when $b = 1$, this theorem gives for the integral

$$(4.3) \quad m_n^* := \int_0^1 x^{a-1} R_n^{(\alpha, \beta)}(x) dx,$$

the first-order recurrence relation

$$\begin{aligned} & (n + \lambda - 1)(n + a)m_n^* + (n + \beta)(n + \lambda - a - 1)m_{n-1}^* \\ & = (2n + \lambda - 1)(\alpha)_n / \Gamma(n + 1). \end{aligned}$$

These recursions can be used to evaluate (4.1) or (4.3), respectively. Note that in [2] (see also [9]), another computational scheme, based on a second-order recurrence relation for a certain auxiliary sequence, was used.

B. *The Wilson polynomials.* The n th Wilson polynomial is defined by [6], [7]

$$(4.4) \quad p_n(t) := (a + b)_n {}_4F_3 \left(\begin{matrix} -n, n + \lambda, a - t^{1/2}, a + t^{1/2} \\ a + b_1, a + b_2, a + b_3 \end{matrix} \middle| 1 \right),$$

$$\lambda = a + b_1 + b_2 + b_3 - 1.$$

In the sequel we use the notation $f(b)$ for $f(b_1)f(b_2)f(b_3)$. Thus, $(a + b)_n$ is short for $(a + b_1)_n(a + b_2)_n(a + b_3)_n$. Notice that the series ${}_4F_3$ in (4.4) is balanced. The Wilson polynomials form an orthogonal system with respect to a positive measure on the real line, provided a, b_1, b_2, b_3 have positive real parts [7]. Application of our theorem yields the three-term recurrence formula

$$\begin{aligned} & (n + \lambda)p_{n+1}(t) - (2n + \lambda) {}_2[t - (n + a)^2 - v(n) + v(n + 1)]p_n(t) \\ & - (n + \lambda - a - b)(2n + \lambda + 1)v(n)p_{n-1}(t) = 0, \\ & v(n) := n(n + a + b - 1)/(2n + \lambda - 1), \end{aligned}$$

which agrees with Wilson's result [6]. For $a + t^{1/2} = 1$ or $a - t^{1/2} = 1$ we have the nonhomogeneous two-term relation

$$(n + \lambda - 1)p_n(t) - np_{n-1}(t) = (a + b - 1)_n(2n + \lambda - 1).$$

C. *The Hahn polynomials.* For real $\alpha > -1, \beta > -1$ and for positive integral N , the Hahn polynomials $Q_n(x) = Q_n(x; \alpha, \beta, N)$ are defined by

$$(4.5) \quad Q_n(x) := {}_3F_2 \left(\begin{matrix} -n, n + \lambda, -x \\ \alpha + 1, -N \end{matrix} \middle| 1 \right) \quad (n = 0, 1, \dots, N),$$

where $\lambda := \alpha + \beta + 1$. The polynomials Q_0, Q_1, \dots, Q_N constitute an orthogonal system on $x = 0, 1, \dots, N$ with respect to the measure

$$(\alpha + 1)_x (\beta + 1)_{N-x} / x! (N - x)!.$$

The three-term recurrence relation for (4.5), obtained earlier by Weber and Erdélyi (see [3]), follows readily by application of the theorem of Section 3.

It is interesting that if $x = -1$ or $x = N + 1$, then the Hahn polynomials satisfy a first-order nonhomogeneous equation. Namely,

$$\begin{aligned} (n + \lambda - 1)(n + \alpha)(n - N - 1)Q_n(-1) + n(n + \beta)(n + \lambda + N)Q_{n-1}(-1) \\ = -\alpha(N + 1)(2n + \lambda - 1), \end{aligned}$$

and, in view of [3],

$$Q_n(N + 1) = (-1)^n \frac{(\beta + 1)_n}{(\alpha + 1)_n} Q_n(-1),$$

a similar result for $Q_n(N + 1)$ holds.

D. *Representation for ${}_3F_2(1)$.* Recently, Wimp [11] has shown that

$${}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \middle| 1 \right),$$

the parameters α_i, β_j being not interrelated, cannot be represented in closed form. Actually, he showed more by proving that the classical Watson formula (see [1, p. 189]; or [4, vol. 1, p. 104]) cannot be generalized. The main tool used in the proof was the difference equation

$$\begin{aligned} (4.6) \quad & \frac{(n + \lambda - 1)(n + a)}{(2n + \lambda - 2)_2} \Phi_{n-1} \\ & - \left[\frac{(n + \beta)(n + a - 1)}{2n + \lambda - 1} - \frac{(n + \beta + 1)(n + a)}{2n + \lambda + 1} + n + b - 1 \right] \Phi_n \\ & - \frac{(n + \alpha + 1)(n + \beta + 1)(n + \lambda - a + 1)}{(n + \lambda)(2n + \lambda + 1)_2} \Phi_{n+1} = 0 \end{aligned}$$

($n + a$ is short for $(n + a_1)(n + a_2)$, etc.), satisfied by

$$C_n := \frac{(a)_n}{(b)_n (n + \lambda)_n} {}_3F_2 \left(\begin{matrix} n + a_1, n + a_2, n + \beta + 1 \\ n + b, 2n + \lambda + 1 \end{matrix} \middle| 1 \right),$$

$$\lambda = \alpha + \beta, \alpha > -1, \beta > -1,$$

which follows by application of the corollary of Section 3. Wimp constructed Eq. (4.6) by using another technique. He also gave other solutions of this equation, corresponding to (3.34)–(3.36).

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